WEAKLY NONLINEAR STATIONARY WAVES IN A THIN FLUID FILM

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UDC 532.51:532.62

Stationary periodic wave flow regimes of thin films of a viscous fluid on an inclined plane are considered.

The intensity of heat and mass transfer processes in film flows, widely spread in energetics, chemical technology, and other areas, depends to a large extent on the flow regime in the film. Therefore, the properties of these flows were investigated theoretically and experimentally in numerous studies. It was discovered that laminar flow with a constant film thickness is unstable with respect to sufficiently long-wave perturbations for film Reynolds numbers exceeding some critical value, which depends on the inclination angle of the sublayer and vanishes for a film on a vertical plate. If the Reynolds number is not too large, the stability loss of non-wave flow leads to establishment of a stationary, periodic regime of weakly nonlinear waves [1]. Theoretically, as shown by numerical calculations [2, 3], stationary periodic solutions of the hydrodynamic equations of a thin fluid layer exist almost in the whole region of wave numbers, corresponding to perturbations unstable in the linear theory. The stability problem of all these solutions remains, however, open, and important experiments indicate the traveling waves are indeed realized with wave numbers in a quite narrow interval around a totally determined value, depending on the Reynolds number of the unperturbed flow and on the properties of the fluid (see [4]). Indeed, the problem arises of determining this a priori unknown value.

According to the linear theory, there exists a unique value of the wave number, corresponding to waves of maximum growth, for which the amplitude increment is maximum. It follows from experiments that this value can also be used for approximate description of weakly nonlinear waves [5, 6], as is also assumed, e.g., in [7]. Generally speaking, however, this assumption does not follow; for unique determination of the wave regime in a nonlinear system one usually uses additional assumptions. Thus, waves were selected in [8] for which energy dissipation is minimal, and an "optimal" regime was considered in [9], corresponding to a minimal film thickness (for a given flow rate). Obviously, a unique nonlinear wave regime, which can be assumed in a certain sense distinct, is the regime with maximum relative wave amplitude, also noted in [8, 9]. This regime, in which the relative amplitude, considered as a function of wave number, reaches a maximum, possesses such properties so that for its nonlinear increment the amplitude growth, also considered as a function of wave number, vanishes together with its first derivative. As follows from the analysis of [10], for a "soft" type of perturbation the instability usually realized in thin films and the weak supercriticality must establish a stationary secondary flow of precisely this nature. For films with a low flow rate this regime was studied in [11] by the small-parameter method, with the parameter taken to be the product of the film Reynolds number by the "long-wave" parameter — the ratio of the unperturbed film thickness to the perturbation wave length. An evolution equation, accurate up to second-order effects (exclusive) in this parameter, was obtained for film waves, a special variant of which coincided with the equation used in [12-15]. Unlike [12-15], however, the validity region of the equations obtained was rigorously indicated in [11]. The wave-number value characterizing the stationary regime of traveling waves was only 1.025 times larger than that calculated in [8], and exceeded approximately 1.178 times the value corresponding to maximum growth waves.

The validity conditions of the results of [11] are quite restricting, and most real film wave flows do not satisfy them. Therefore, in the present study the evolution equation and the characteristics of the stationary wave regime for low flow rates have been

A. M. Gor'kii Ural State University, Sverdlovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 45, No. 4, pp. 566-576, October, 1983. Original article submitted October 12, 1982.

 0022-0841/83/4504-1112$07.50

obtained accurately to second-order effects in the small parameter indicated above. Besides, the method used of studying the stationary regime is also valid within the approximate Karman–Pollhausen approach (approximating the instantaneous velocity profile in the film by a self-similar polynomial gives fair results for two-dimensional waves [1, 4]), which made it possible to advance to the region of significantly higher rates.

We introduce dimensionless variables and parameters (the primes denote the corresponding dimensional variables)

$$t=\frac{u_0}{\lambda}t',\ x=\frac{x'}{\lambda},\ y=\frac{y'}{h_0},\ \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}=\frac{1}{u_0}\begin{Bmatrix} v_x' \\ v_y' \end{Bmatrix},$$
$$\varphi=\frac{h-h_0}{h_0},\ p=\frac{\mathrm{Re}}{\rho u_0^2}p',\ \mathrm{Re}=\frac{u_0h_0}{\nu},\ T=\frac{3\varepsilon^3\mathrm{We}}{\cos\alpha},\tag{1}$$
$$\mathrm{We}=\frac{\sigma}{\rho g h_0^2},\ \varepsilon=\frac{h_0}{\lambda},\ u_0=\left(\frac{\cos\alpha}{3}\frac{g}{\nu}\right)^{1/3}Q^{2/3},\ h_0=\left(\frac{3}{\cos\alpha}\frac{\nu Q}{g}\right)^{1/3},$$

where u_0 and h_0 are the mean velocity and film thickness in the unperturbed (non-wave) regime.

The equations of motion in the thin fluid layer approximation with the associated boundary conditions in the variables (1) were similarly treated in [11], as well as in a number of other studies not cited here.

The solution of this problem is represented in the form

$$v_x=\sum_{n=0}^{\infty}\varepsilon^n v_x^{(n)},\ v_y=\sum_{n=0}^{\infty}\varepsilon^n v_y^{(n)},\ p=\sum_{n=-1}^{\infty}\varepsilon^n p_n.\tag{2}$$

Linear problems can be formulated for the coefficients of series (2), formally obtained within the various orders of powers of ε. In this case we assume that $\varepsilon\mathrm{Re}$ equals in order of magnitude the relative perturbation amplitude of the film thickness or less (unlike [11], where $\varepsilon\mathrm{Re}\leqslant\varphi^2$). Due to the awkwardness, we write only the expressions for $v_x^{(i)}$, following from solving the problems indicated:

$$v_x^{(0)}=\left(3+T\frac{\partial^3\varphi}{\partial x^3}\right)\left(1+\varphi-\frac{y}{2}\right)y,$$
$$v_x^{(1)}=-\left[3\,\mathrm{tg}\,\alpha\frac{\partial\varphi}{\partial x}\eta+\mathrm{Re}\sum_{i=2}^{5}\frac{1}{i}V_i\eta^i\right]y+\frac{3}{2}\mathrm{tg}\,\alpha\frac{\partial\varphi}{\partial x}y^2+\mathrm{Re}\sum_{i=2}^{5}\frac{1}{i(i+1)}V_iy^{i+1},$$
$$v_x^{(2)}=-\mathrm{Re}\left[3\,\mathrm{tg}\,\alpha\sum_{i=1}^{4}\frac{1}{i(i+1)}U_i\eta^i+\mathrm{Re}\sum_{i=1}^{8}\frac{1}{i(i+1)}W_i\eta^{i+1}\right]y-$$
$$-\sum_{i=0}^{2}\frac{1}{(i+1)^2}N_i\eta^{i+1}y+N_*y+\mathrm{Re}\left[3\,\mathrm{tg}\,\alpha\sum_{i=1}^{4}\frac{1}{i(i+1)(i+2)}U_iy^{i+2}+\right.\tag{3}$$
$$\left.+\mathrm{Re}\sum_{i=1}^{8}\frac{1}{i(i+1)(i+2)}W_iy^{i+2}\right]+\sum_{i=0}^{2}\frac{1}{(i+1)^2(i+2)}N_iy^{i+2}.$$

Here

$$V_0=\frac{1}{2}T\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2};\ V_1=3+T\frac{\partial^3\varphi}{\partial x^3};\ V_2=\frac{\partial F}{\partial t};$$
$$V_3=\frac{1}{2}\left(H\frac{\partial F}{\partial x}-T\frac{\partial^4\varphi}{\partial t\partial x^3}\right);\ V_4=-\frac{1}{3}TG;\ V_5=\frac{1}{12\eta}TG;\ \eta=1+\varphi;$$

$$F = 3\varphi + T\frac{\partial^3\varphi}{\partial x^3}\eta;\ H = \left(3 + T\frac{\partial^3\varphi}{\partial x^3}\right)\eta;\ G = H\frac{\partial^4\varphi}{\partial x^4};$$

$$U_1 = -\frac{\partial}{\partial t}\left(\eta\frac{\partial\varphi}{\partial x}\right);\ U_2 = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t} - V_1\eta^2\frac{\partial\varphi}{\partial x}\right);$$

$$U_3 = \eta\frac{\partial}{\partial x}\left(V_1\frac{\partial\varphi}{\partial x}\right);\ U_4 = -\frac{1}{3}\frac{\partial}{\partial x}\left(V_1\frac{\partial\varphi}{\partial x}\right);$$

$$W_1 = -\frac{\partial}{\partial t}\left(\sum_{i=2}^{5}\frac{1}{i}V_i\eta^i\right);\ W_2 = -\frac{\partial}{\partial x}\left(V_1\sum_{i=2}^{5}\frac{1}{i}V_i\eta^{i+1}\right);$$

$$W_3 = \frac{1}{2}\frac{\partial V_2}{\partial t} + \frac{\partial V_1}{\partial x}\sum_{i=2}^{5}\frac{1}{i}V_i\eta^i;$$

$$W_4 = \frac{1}{3}\frac{\partial V_3}{\partial t} + \frac{1}{2}V_1\frac{\partial V_2}{\partial x}\eta - \frac{1}{3}\frac{\partial}{\partial x}(V_1\eta)V_2; \tag{4}$$

$$W_5 = \frac{1}{4}\frac{\partial V_4}{\partial t} + \frac{1}{3}V_1\frac{\partial V_3}{\partial x}\eta - \frac{5}{24}V_1\frac{\partial V_2}{\partial x} - \frac{5}{12}\frac{\partial}{\partial x}(V_1\eta)V_3;$$

$$W_6 = \frac{1}{5}\frac{\partial V_5}{\partial t} + \frac{1}{4}V_1\frac{\partial V_4}{\partial x}\eta - \frac{3}{20}V_1\frac{\partial V_3}{\partial x} - \frac{9}{20}\frac{\partial}{\partial x}(V_1\eta)V_4 + \frac{1}{12}V_3\frac{\partial V_1}{\partial x};$$

$$W_7 = \frac{1}{5}V_1\frac{\partial V_5}{\partial x}\eta - \frac{7}{60}V_1\frac{\partial V_4}{\partial x} - \frac{7}{15}\frac{\partial}{\partial x}(V_1\eta)V_5 + \frac{7}{60}V_4\frac{\partial V_1}{\partial x};$$

$$W_8 = -\frac{2}{21}V_1\frac{\partial V_5}{\partial x} + \frac{2}{15}V_5\frac{\partial V_1}{\partial x},$$

$$N_1 = -4\frac{\partial^2}{\partial x^2}(V_1\eta);\ N_2 = 3\frac{\partial^2 V_1}{\partial x^2};\ N_* = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(V_1\eta)\eta^2\right] - \frac{1}{6}\frac{\partial}{\partial x}\left(\eta_3\frac{\partial V_1}{\partial x}\right) + \left[4\frac{\partial}{\partial x}(V_1\eta)\eta - 2\frac{\partial V_1}{\partial x}\eta^2\right]\frac{\partial\varphi}{\partial x};$$

$$N_0 = \frac{\partial V_0}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x}\eta^2\right) - \frac{\partial}{\partial x}\left[\eta\frac{\partial}{\partial x}(V_1\eta)\right].$$

In deriving (3), (4) it was assumed that $T \leqslant 1$ (the surface tension can be quite large) and $\varepsilon\,\mathrm{tg}\,\alpha \leqslant 1$ (the planar sublayer can be quite nearly horizontal).

It follows from the continuity equation that

$$\frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^\eta v_x dy\right) = 0. \tag{5}$$

Using expressions (3) and (4), we obtain from (5) an evolution equation

$$\frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial x}\left[\left(1 + \frac{1}{3}T\frac{\partial^3\varphi}{\partial x^3}\right)\eta^3\right] - \varepsilon\frac{\partial}{\partial x}\left[\mathrm{tg}\,\alpha\frac{\partial\varphi}{\partial x}\eta^3 + \right. \tag{6}$$

$$\left. + \mathrm{Re}\left(\frac{5}{24}V_2 + \frac{3}{20}V_3\eta + \frac{7}{60}V_4\eta^2 + \frac{2}{21}V_5\eta^3\right)\eta^4\right] +$$

$$+ \varepsilon^2\frac{\partial}{\partial x}\left\{\frac{1}{2}N_*\eta^2 + \mathrm{Re}\left[3\,\mathrm{tg}\,\alpha\sum_{i=1}^{4}m_iU_i\eta^{i+3} + \mathrm{Re}\sum_{i=1}^{8}m_iW_i\eta^{i+3}\right] + \sum_{i=0}^{2}n_iN_i\eta^{i+3}\right\} = 0,$$

where

$$m_i = \frac{2-(i+2)(i+3)}{2i(i+1)(i+2)(i+3)};\quad n_i = \frac{2-(i+2)(i+3)}{2(i+1)^2(i+2)(i+3)}. \tag{7}$$

We stress that no restrictions are imposed on the quantity φ in (6). In the first approximation in ε, Eq. (6) coincides with the equation obtained in [11].

Below we consider in detail weakly nonlinear waves ($\varphi \ll 1$) when $T \leqslant \varepsilon \ll 1$ and $\operatorname{tg}\alpha \leqslant 1$ (i.e., $\varepsilon \operatorname{tg}\alpha \leqslant \varepsilon \ll 1$), so that Eq. (6) can be simplified substantially. For this we take into account that in the zeroth approximation in ε and the first in φ, Eq. (6) transforms into a well-known linear equation, whose solution can be an arbitrary function $f(x - 3t)$, and according to standard methods we replace the derivative $\partial/\partial t$ in terms of order ε^2 in (6) by $-3\partial/\partial x$. In addition, we choose the longitudinal scale λ equal to h_0, so that in the dimensionless coordinate system thus determined $\varepsilon = 1$, but in return the dimensionless wavelength L must be much larger than unity. With account of (4), (7) we then obtain from (6) after some calculations

$$\frac{\partial\varphi}{\partial t} + 3(1+\varphi)^2\frac{\partial\varphi}{\partial x} + \frac{\partial}{\partial x}\left[(1+3\varphi)\left(B\frac{\partial^3\varphi}{\partial x^3} - \operatorname{tg}\alpha\frac{\partial\varphi}{\partial x}\right)\right] - \frac{5}{8}\operatorname{Re}\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t} + \frac{27}{25}\frac{\partial\varphi}{\partial x}\right) + \frac{69}{20}\operatorname{Re}\frac{\partial}{\partial x}\left(\varphi\frac{\partial\varphi}{\partial x}\right) +$$

$$+\frac{45}{56}B\operatorname{Re}\frac{\partial^5\varphi}{\partial x^5} + \left(3 + \frac{27}{28}\operatorname{Re}^2 - \frac{45}{56}\operatorname{Re}\operatorname{tg}\alpha\right)\frac{\partial^3\varphi}{\partial x^3} = 0, \quad B = \frac{\mathrm{We}}{\cos\alpha}. \tag{8}$$

This equation was written down with an accuracy up to terms of order φ^3, inclusive; this is sufficient for analyzing weakly nonlinear stationary wave regimes [10, 11].

We represent the unknown φ of Eq. (8) in the form

$$\varphi = \sum_{n=-\infty}^{\infty} \Phi_n \exp[in(\omega t - kx)], \quad \omega = \Omega - i\gamma, \quad k = 2\pi\frac{1}{L}, \tag{9}$$

where k, Ω, and γ are real quantities. From the reality condition of φ it follows that $\Phi_{-n} = \Phi_n^*$, where the asterisk denotes complex conjugation. For weakly nonlinear periodic waves it can be assumed that the amplitude of the first harmonic (n = 1) is much larger than the amplitudes of all other harmonics. Substituting (9) into Eq. (8), and restricting ourselves to first-order terms in the quantity $q = \Phi_1\Phi_{-1} = |\Phi_1|^2$, we have the following dispersion relation, corresponding to harmonics with n = 1 and n = 2:

$$\begin{gathered}[i(\omega - 3k) + A_1k^2 + Bk^4 - C_1\omega k - iC_2k^5 + iDk^3]\Phi_1 + \\ + (A_2k^2 - 6ik)(\Phi_1\Phi_0 + \Phi_{-1}\Phi_2) + 3Bk^4(7\Phi_{-1}\Phi_2 + \Phi_1\Phi_0) - 3ik\Phi_1^2\Phi_{-1} = 0,\end{gathered} \tag{10}$$

$$[i(\omega - 3k) + 2A_1k^2 + 8Bk^4 - 2C_1\omega k - 16iC_2k^5 + 4iDk^3]\Phi_2 = (3ik - 3Bk^4 - A_2k^2)\Phi_1^2.$$

Here

$$\begin{gathered}A_1 = \operatorname{tg}\alpha + \frac{27}{40}\operatorname{Re}; \quad A_2 = 3\left(\operatorname{tg}\alpha - \frac{23}{20}\operatorname{Re}\right); \quad C_1 = \frac{5}{8}\operatorname{Re}; \\ C_2 = \frac{45}{56}B\operatorname{Re}; \quad D = 3\left(1 + \frac{9}{28}\operatorname{Re}^2 - \frac{15}{56}\operatorname{Re}\operatorname{tg}\alpha\right).\end{gathered} \tag{11}$$

It follows from (10) that

$$\Phi_2 = \frac{1}{2}\,\frac{6ik - 3Bk^4 - A_2k^2}{i(\omega - 3k - 16C_2k^5 + 4Dk^3) - 2C_1\omega k + 2A_1k^2 + 8Bk^4} \tag{12}$$

and, further,

$$\begin{gathered}i(\omega - 3k - C_2k^5 + Dk^3) - C_1\omega k + A_1k^2 + Bk^4 + (-6ik + A_2k^2 + 3Bk^4)\Phi_0 + \left(-3ik + \frac{1}{2}\times\right. \\ \left.\times\frac{(6ik - 3Bk^4 - A_2k^2)(-6ik + A_2k^2 + 21Bk^4)}{i(\omega - 3k - 16C_2k^5 + 4Dk^3) - 2C_1\omega k + 2A_1k^2 + 8Bk^4}\right)q = 0.\end{gathered} \tag{13}$$

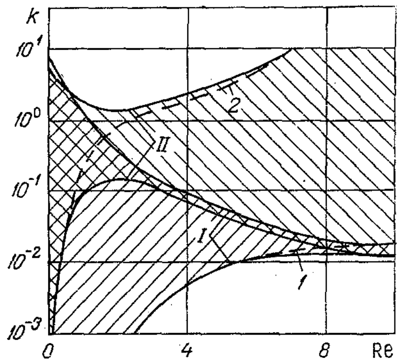

Fig. 1

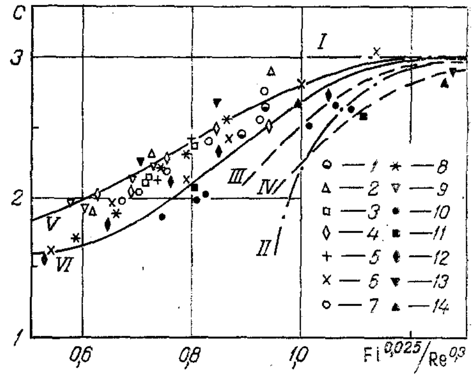

Fig. 2

Fig. 1. Neutral stability curves for flow over a vertical plate according to [11] (curves 1, 2 are for water, R = 11.93, and transformer oil, R = 1.79; stability regions are above curves) and by Eq. (14) (I and II correspond to water and oil, and the stability regions between the curves are shaded).

Fig. 2. Phase velocity of waves. Curves obtained from theory: I) [11]; III) (13); V) (21)-(24) at q = 0 (for waves of maximum growth); II, IV, VI, the same for $q \neq 0$ (for stationary nonlinear waves). Points obtained by experiment: 1) water, R = 11.34; 2) aqueous solutions of ethyl alcohol and glycerin, R = 8.25; 3) ethyl alcohol, R = 6.40; 4-9) aqueous solution of glycerin for R = 8.15, 6.30, 4.77, 3.98, 4.77, and 8.10, respectively [19]; 10) water, R = 10.94; 11) ethyl alcohol, R = 6.60 [16]; 12) water, R = 11.93 [17]; 13, 14) transformer oil at R = 1.79, 1.16 [18], $\alpha = 0$.

It follows from (12) that $\Phi_2 \sim q$. The quantity Φ_0 is determined from the requirement that the dimensionless flow rate in the film be equal to its given unperturbed value, i.e., to unity in the variables (1). It hence follows, as in [11], that $\Phi_0 = -2q$, which finally determines Eq. (13). The harmonics in (9) with n > 2 are of higher order of smallness in q: $\Phi_n \sim q^{n/2}$ [10], so that they can be neglected within the accuracy chosen.

Equation (13) describes Ω and γ as functions of k and q. Within the linear theory ($q \to 0$) we obtain from (13) the dependence of Ω and γ on the wave number, determining the neutral stability curve of unperturbed flow, as well as the frequency, velocity, and increment (or damping decrement) of waves of various lengths. For q = 0 we have

$$\Omega = \Omega_1(k) = \frac{3 + (A_1C_1 - D)k^2 + (C_2 + BC_1)k^4}{1 + C_1^2k^2} k, \qquad (14)$$
$$\gamma = \Gamma_1(k) = \frac{(3C_1 - A_1)k^2 - (DC_1 + B)k^4 + C_1C_2k^6}{1 + C_1^2k^2}.$$

Analysis of the second relation (14) shows that flow of a film with a constant thickness is unstable not only with respect to long-wave perturbations, as is well known from numerous preceding studies, but also with respect to perturbations with sufficiently short wavelengths. The instability regions noted are separated by intermediate stability regions, shown in Fig. 1 for waves on a vertical plate. Thus, the analysis provided reflects not only mechanisms of appearance of nearly harmonic waves, but also determines the degree and excitation mechanisms of rolling waves. As is well known, the occurrence of a small shortwave ripple at almost-harmonic wavelengths was observed experimentally many times.

For $q \neq 0$, Ω is conveniently represented in the form

$$\Omega = \Omega_1(k) + q\Omega_2(k), \quad \gamma = \Gamma_1(k) + q\Gamma_2(k). \qquad (15)$$

Taking into account that the stationary regime corresponds to a vanishing amplitude increment, while this value must correspond to the maximum value of γ, considered as a function of k for fixed q, we obtain an equation for the determination of k and q characterizing the stationary wave regime:*

$$\Gamma_1(k) + q\Gamma_2(k) = 0, \quad \frac{d\Gamma_1}{dk} + q\frac{d\Gamma_2}{dk} = 0. \tag{16}$$

These quantities can also be expressed within the accuracy adopted in terms of the wave number k_m of maximum growth waves. From (16) we have

$$q = -\frac{\Gamma_1(k_m)}{\Gamma_2(k_m)}, \quad k = k_m + \frac{\Gamma_1(k_m)}{\Gamma_2(k_m)}\frac{d\Gamma_2}{dk_m}\left(\frac{d^2\Gamma_1}{dk_m^2}\right)^{-1}. \tag{17}$$

Figures 2-4 show results of calculating the quantities q and k, as well as the phase velocity of stationary traveling waves according to the nonlinear theory considered above and the theory of [11]. Also shown are curves obtained from the corresponding linear variants of the theory for waves of maximum growth, as well as several experimental data. Unfortunately, for small Reynolds numbers, for which the theory suggested is valid, there exist few experimental data, while their accuracy is not high. The latter is due to difficulties of measuring wave characteristics at low wave amplitudes. In this region, therefore, one cannot talk convincingly about good agreement of theoretical results with experiment.

Most known experiments refer to the region Re $\gtrsim \varepsilon^{-1}$, to which the small-parameter method is not applicable. (Obviously, retention of terms of order ε^3 and higher in series (2) cannot lead, in principle, to extension of the results obtained to the region indicated.) For approximate analysis of the stationary wave regimes in this region, we use below a method of integral relations in the form suggested in [7]. The instantaneous longitudinal velocity distribution over the thickness of the wavy film is approximated by the self-similar parabolic profile

$$v_x = \frac{3}{2}(1+\psi)\left(2-\frac{y}{\eta}\right)\frac{y}{\eta}, \tag{18}$$

where ψ is treated as an unknown variable in addition to φ. Comparison of (18) with (3) shows that the parabolic profile describes the velocity distribution inaccurately even for small Re. But comparison with experimental data (see, e.g., [4]) shows convincingly that the error of this approximation is not very large until moderately high values of Re.

Following transformations similar to [7], from the hydrodynamic equations of a thin layer of a viscous fluid we obtain a system of equations for φ and ψ, which in the accepted notation is written for $\varepsilon = 1$ in the following form:

$$\begin{gathered}\mathrm{Re}\left\{\frac{\partial}{\partial t}[(1+\psi)(1+\varphi)] + \frac{6}{5}\frac{\partial}{\partial x}[(1+\psi)^2(1+\varphi)]\right\} = -\frac{3(1+\psi)}{1+\varphi} - \\ -3\,\mathrm{tg}\,\alpha(1+\varphi)\frac{\partial\varphi}{\partial x} + T(1+\varphi)\frac{\partial^3\varphi}{\partial x^3} + 3(1+\varphi), \\ \frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial x}[(1+\psi)(1+\varphi)] = 0, \quad \int_0^\eta v_x dy = (1+\psi)(1+\varphi).\end{gathered} \tag{19}$$

We assume that $\varphi \ll 1$, $\psi \ll 1$, and write in addition to (9)

$$\psi = \sum_{n=-\infty}^{\infty} \Psi_n \exp[in(\omega t - kx)], \quad \Psi_{-n} = \Psi_n^*. \tag{20}$$

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*We note that the requirement of maximum value of γ as function of wave number k is not an additional hypothesis. The meaning of this requirement is discussed in [10, 11], and consists of the following. On one hand, if the wave regime is characterized by a wave number k_s, the stationarity condition of this regime requires vanishing of the corresponding oscillation increment, i.e., $\gamma(k_s) = 0$. On the other hand, the regime under consideration corresponds to almost-harmonic oscillations. This implies that oscillations corresponding to all other k values must be damped, i.e., $\gamma(k) < 0$ for $k \neq k_s$. Obviously, simultaneous satisfaction of these requirements is possible only if the function $\gamma(k)$ reaches a maximum value at the point $k = k_s$, while this value vanishes exactly.

Substituting (9) and (20) into (19), we obtain instead of (10)

$$
\begin{gathered}
(E+3)\Psi_1+(E_1-6)\Phi_1+(E-3)(\Psi_{-1}\Phi_2+\Psi_0\Phi_1+\Psi_1\Phi_0+\\
+\Psi_2\Phi_{-1})-E_2(2\Psi_{-1}\Psi_2+2\Psi_0\Psi_1+\Psi_1^2\Phi_{-1}+2\Psi_{-1}\Psi_1\Phi_1)=\\
=-6\Phi_{-1}\Phi_2-6\Phi_0\Phi_1+9\Phi_1^2\Phi_{-1}-3\Psi_{-1}\Phi_1^2-6\Phi_{-1}\Phi_1\Psi_1+\\
+(3ik\,\mathrm{tg}\,\alpha+iTk^3)\Phi_1+3ik\,\mathrm{tg}\,\alpha(\Phi_0\Phi_1+\Phi_{-1}\Phi_2)+iTk^3(7\Phi_{-1}\Phi_2+\Phi_0\Phi_1),
\end{gathered}
\tag{21}
$$

$$
\begin{gathered}
(2E+3)\Psi_2+(2E_1-6)\Phi_2+(2E-3)\Psi_1\Phi_1-2E_2\Psi_1^2=\\
=(6ik\,\mathrm{tg}\,\alpha+8iTk^3)\Phi_2+(-3+3ik\,\mathrm{tg}\,\alpha+iTk^3)\Phi_1^2,\\
-2\Phi_0+2\Phi_{-1}\Phi_1+\Psi_0-\Psi_{-1}\Phi_1-\Psi_1\Phi_{-1}=0,\\
i\omega\Phi_2-ik(\Psi_2+\Phi_2+\Psi_1\Phi_1)=0,\\
i\omega\Phi_1-ik(\Psi_1+\Phi_1+\Psi_{-1}\Phi_2+\Psi_0\Phi_1+\Psi_1\Phi_0+\Psi_2\Phi_{-1})=0,
\end{gathered}
$$

where

$$
E=i\,\mathrm{Re}\left(\omega-\frac{12}{5}k\right);\ E_1=i\,\mathrm{Re}\left(\omega-\frac{6}{5}k\right);\ E_2=\frac{6}{5}ik\,\mathrm{Re}. \tag{22}
$$

By the equality condition of the dimensionless flow rate to unity (see last equation in (19)) we get

$$
\Psi_0+\Phi_0+\Psi_{-1}\Phi_1+\Psi_1\Phi_{-1}=0. \tag{23}
$$

It follows from (22) and (23) that

$$
\Phi_0=2\left(1-\frac{2}{3}\frac{\Omega}{k}\right)q,\ \Psi_0=-\frac{2}{3}\frac{\Omega}{k}q, \tag{24}
$$

$$
\Psi_1=\frac{\omega-k}{k}\Phi_1-\left[\frac{\omega^*-k}{k}F_1+F_2+2\frac{\omega-k}{k}\left(1-\frac{2}{3}\frac{\Omega}{k}\right)-\frac{2}{3}\frac{\Omega}{k}\right]\Phi_1^3,\ \Phi_2=F_1q,\ \Psi_2=F_2q,
$$

$$
F_1=\frac{2E_2k^{-2}(\omega-k)^2+6k^{-1}(\omega-k)-3+3ik\,\mathrm{tg}\,\alpha+iTk^3}{(2E+3)k^{-1}(\omega-k)+2E_1-6-6ik\,\mathrm{tg}\,\alpha-8iTk^3},
$$

$$
F_2=\frac{2E_2k^{-2}(\omega-k)^2-(2E-3)k^{-1}(\omega-k)-2E_1+3+9ik\mathrm{tg}\alpha+9iTk^3}{(2E+3)k^{-1}(\omega-k)+2E_1-6-6ik\mathrm{tg}\alpha-8iTk^3}k^{-1}(\omega-k).
$$

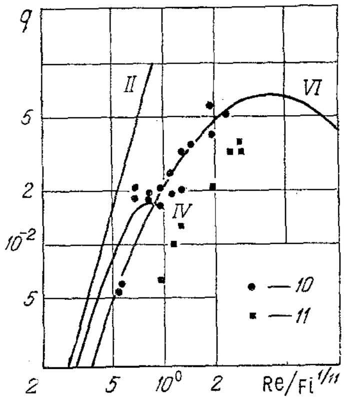
Relations (24) completely determine the first equation in (21), from which follows a dispersion equation relating Ω and γ with k and $q = \Phi_1\Phi_{-1}$, completely in analogy with (13). To analyze the latter equation it is convenient to use expressions (15)-(17). In the linear approximation the dispersion relation of the problem considered was investigated in [7], and therefore we do not write down the relation similar to (14).

The dependences of the quantity q, the wave number, the velocity of linear waves of maximum growth, and the nonlinear stationary waves, corresponding to the theory discussed, on the physical and regime parameters are shown in Figs. 2-4. It is seen that they agree not badly with the experimental data of [16-19].

Thus, unlike the known studies of wave flows of thin layers of a viscous fluid, the stationary almost-harmonic regime of traveling waves is completely determined in sufficiently wide intervals of the Reynolds number and ordinarily used "film number"

$$
\mathrm{Fi}=R^{10}=9\mathrm{Re}^2\mathrm{We}^3=\frac{\sigma^3}{\rho^3g\nu^4},\ R=\frac{\sigma^{0,3}}{\rho^{0,3}g^{0,1}\nu^{0,4}} \tag{25}
$$

without including any further considerations, which is also a principal result of the present study. The stationary regime parameters have been determined with an accuracy up to components of order q, the square of the relative amplitude of the main harmonic perturbation of the film thickness. We note that enhancement of accuracy of the dispersion relations



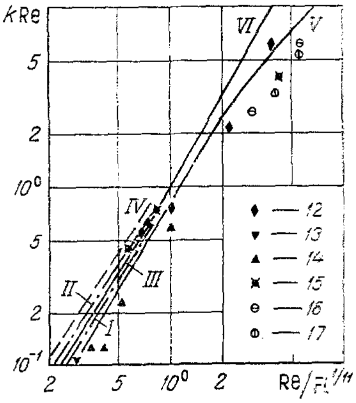


Fig. 3 Fig. 4

Fig. 3. Wave amplitude. Notation same as in Fig. 2; $\alpha = 0$.

Fig. 4. The wave number of waves: 15) water, R = 11.09 [5]; 16) ethyl alcohol, R = 6.40 [7]; 17) aqueous solution of glycerin, R = 7.71 [7]; remaining notation same as in Fig. 2; $\alpha = 0$.

obtained only by inclusion of harmonics with $|n| > 2$ in the series (9) and (20) is not possible, since the evolution equation (8) was written accurately up to terms of order $\varphi^3 \sim q^{3/2}$ (but not $\varphi^4 \sim q^2$), while approximation (18) is inaccurate to start with.

The results obtained also make it possible to write down several approximate analytic relations for the various characteristics of stationary wave regimes, generalizing the results of [11]. These relations, however, are quite awkward and are not given here due to lack of space. In many cases it is natural to resort to numerical calculations, as was done above. This problem will be considered in more detail in the near future in analyzing heat and mass transfer processes in thin films containing wave motion.

NOTATION

A_i, C_i, D, quantities introduced in (11); B, quantity introduced in (8); c, dimensionless phase velocity; E, E_i, quantities introduced in (22); F, H, G, functions introduced in (4); F_i, quantities introduced in (24); g, gravity acceleration; h, h_0, film thicknesses in the wave and non-wave regimes; k, wave number; k_m, wave number for waves of maximum growth; L, dimensionless wavelength; m_i, n_i, parameters in (7); N_i, N_*, functions introduced in (4); p, p', dimensionless and dimensional pressures; q, square of the main harmonic of the wave; U_i, V_i, W_i, functions introduced in (4); T, parameter in (1); t, t', dimensionless and dimensional times; u_0, mean velocity in the non-wave regime; v, v', dimensionless and dimensional velocities; x, y, x', y', dimensionless and dimensional longitudinal and transverse coordinates; α, angle between the sublayer film and the vertical planes; Γ_i, quantities introduced in (15); γ, amplitude increment; ε, "long-wave" parameter; η, dimensionless film thickness; λ, linear longitudinal scale; γ, kinematic viscosity; ρ, fluid density; σ, surface tension coefficient; Φ_i, functions introduced in (9); φ, dimensionless wave amplitude; Ψ_i, functions introduced in (20); ψ, dimensionless velocity perturbation introduced in (18); Ω, wave frequency; Ω_i, quantities introduced in (15); ω, complex wave frequency; Fi, film number; Re, We, Reynolds and Weber numbers; asterisk denotes complex conjugation.

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INTERFEROMETER INVESTIGATION OF CONVECTION IN A HORIZONTAL FLUID LAYER

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UDC 532.529.2:532.574.7

The temperature field in free convective motion of a non-Newtonian fluid is studied by using an interferometer. A method of constructing the flow pattern by means of the interferograms obtained is developed.

An interferometer method is used extensively to visualize temperature fields in gases and liquids [1]. Thus, thermal regime characteristics are determined for a horizontal liquid layer heated from below. A detailed description of the test and analysis methods of the results obtained is presented in [2-4].

The flow pattern in gases and liquids can be observed by using an interferometer only when the velocity changes in the domain under investigation are large, resulting in noticeable density changes (compressibility) and, therefore, in changes in the refractive index also. The velocity gradients in free convection in a horizontal layer are so small that it is impossible to observe the flow pattern by means of an interferometer.

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 0022-0841/83/4504-1120$07.50